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19 REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR TR-78-1102	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) VARIABLE METRIC METHODS AND FILTERING THEORY.	5. TYPE OF REPORT & PERIOD COVERED Interim rept.	
6. AUTHOR(s) Sanjoy K. Mitter and Pal Toldalagi	7. PERFORMING ORG. REPORT NUMBER ESL-P-826	8. CONTRACT OR GRANT NUMBER(s) AFOSR-77-3281
9. PERFORMING ORGANIZATION NAME AND ADDRESS Massachusetts Institute of Technology Electronic Systems Laboratory Cambridge, Massachusetts 02139	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F / 2304 / A1	
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332	12. REPORT DATE June 1978	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 129p.	13. NUMBER OF PAGES 8	
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of abstract entered in Block 20, if different from Report)	DDC RECEIVED JUN 30 1978 F	
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In this paper we show that there is a close relationship between variable metric methods of function minimization and filtering of linear stochastic systems with disturbances which are emodelled as unknown but bounded functions. We develop new variable metric algorithms for function minimization.		

## VARIABLE METRIC METHODS AND FILTERING THEORY

AFOSR-TR- 78-1102

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## ABSTRACT

In this paper we show that there is a close relationship between variable metric methods of function minimization and filtering of linear stochastic systems with disturbances which are modelled as unknown but bounded functions. We develop new variable metric algorithms for function minimization.

1. INTRODUCTION

The objective of this paper is to show that there is a close relationship between variable metric methods of function minimization and filtering of linear stochastic systems with disturbances which are modelled as unknown but bounded functions.

It is well known that Newton's method for function minimization exhibits quadratic convergence in the neighborhood of the minimum. This rapid convergence rate however is obtained at the expense of requiring second derivative computations and solution of a linear equation at each iteration stage. On the other hand, variable metric methods do not require second derivative computations nor matrix inversion (solution of a linear equation) and versions of this algorithm are known to exhibit reasonably rapid convergence. Intuitively, one may consider a variable metric method as one where an estimate of the Hessian (or inverse of a Hessian) is obtained on the basis of information on function values and gradient values in past iterations and the next step is determined on the basis of this estimate. In this paper, we attempt to make this intuitive notion precise.

The work closest in spirit to this work is the doctoral dissertation of THOMAS [4]. The stochastic models we derive are however, somewhat different and we exploit linear filtering theory to the fullest extent possible. We obtain algorithms which do not require accurate line search algorithms as was also done by Thomas.

2. FILTERING MODEL FOR THE ALGORITHM

Consider the problem of minimizing

$$\{f(x) | x \in \mathbb{R}^n\}, \text{ where}$$

(2.1)

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$f$  is assumed to be thrice continuously differentiable on  $\mathbb{R}^n$ .

Let

$$\nabla f(x) = g(x) \quad \text{and} \quad D^2 f(x) = G(x). \quad (2.2)$$

Let  $x^*$  be a local minimum of  $f$  and in some open, convex neighborhood  $D$  of  $x^*$ , let us assume

$$\|G(x_k + \theta_1 s_k) - G(x_k + \theta_2 s_k)\| \leq L|\theta_1 - \theta_2| \cdot \|s_k\|, \text{ where } L > 0 \quad (2.3)$$

for all  $x_k, x_k + s_k \in D$ , all  $\theta_1, \theta_2 \in [0, 1]$ .

We wish to discuss iterative algorithms for minimizing  $f(x)$  and the algorithm proceeds as  $x_{k+1} = x_k + s_k$ ,  $k = 0, 1, 2, \dots$ .

Let us use the notation

$$\begin{aligned} G_k(\theta) &= G(x_k + \theta s_k) \\ g_k(\theta) &= g(x_k + \theta s_k), \quad k = 0, 1, 2, \dots \end{aligned} \quad (2.4)$$

It is easy to see that there exists  $U_k \in L^1(0, 1; \mathcal{L}(\mathbb{R}^n))$  such that

$$\left. \begin{aligned} G_k(\theta) - G_k(0) &= \int_0^\theta U_k(t) dt, \text{ with} \\ \|U_k(\theta)\| &\leq L\|s_k\|, \quad k = 0, 1, 2, \dots, \theta \in [0, 1] \end{aligned} \right\} \quad (2.5)$$

Also

$$g_k(\theta) = g_k(0) + \int_0^\theta G_k(t) s_k dt. \quad (2.6)$$

Evaluating (2.5) and (2.6) at  $\theta = 1$ , and using the natural notation  $G_k(1) = G_{k+1}$ ,  $G_k(0) = G_k$ ,  $g_k(1) = g_{k+1}$ , etc., we get

$$\left. \begin{aligned} G_{k+1} &= G_k + \int_0^1 U_k(t) dt \\ g_{k+1} &= g_k + G_k s_k + \int_0^1 [G_k(t) - G_k(0)] s_k dt \end{aligned} \right\} \quad (2.8)$$

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Let

$$v_k = \int_0^1 u_k(t) dt$$

$$w_k = \int_0^1 [G_k(t) - G_k(0)] s_k dt$$

Then, we may rewrite (2.8) as

$$\left. \begin{aligned} G_{k+1} &= G_k + v_k \\ g_{k+1} &= g_k + G_k s_k + w_k \end{aligned} \right\} \quad (2.9)$$

It is natural to think of  $v_k$  and  $w_k$  as process and observation noise respectively. They are obviously correlated. We now attempt to bound the noise.

### 3. BOUNDS ON THE NOISE

To do the bounding, we use the following device: Let  $\gamma_i$  denote the  $i^{\text{th}}$  row of  $G$ . We then use the isomorphism

$$i : \mathcal{L}(\mathbb{R}^n) \rightarrow \mathbb{R}^{n^2} : G \mapsto \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}.$$

We can then rewrite equation (2.8) in differential form:

$$\left. \begin{aligned} \frac{d}{d\theta} (iG_k) &= iU_k(\theta) \\ \frac{d}{d\theta} g_k(\theta) &= (I_n \otimes s_k') (iG_k(\theta)) \end{aligned} \right\} \quad (2.10)$$

In the above ' denotes transpose and  $\otimes$  denotes tensor product. Writing (2.10) in vector-matrix form:

$$\frac{d}{d\theta} \begin{bmatrix} iG_k(\theta) \\ g_k(\theta) \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ I_n \otimes s_k' & 0 \end{pmatrix} \begin{pmatrix} iG_k(\theta) \\ g_k(\theta) \end{pmatrix} + \begin{pmatrix} iU_k(\theta) \\ 0 \end{pmatrix} \quad (2.11)$$

We are interested in bounding  $v_k$  and  $w_k$  as  $U_k(\cdot)$  varies over the class of all mappings given by (2.5). Clearly, the set of all  $(iV_k, w_k)$  as  $U_k(\cdot)$  varies is a convex set in  $\mathbb{R}^{n^2} + \mathbb{R}^n$ . Let  $\Omega_k$  denote the set. We can compute the support function of this set and estimate that the support function  $\eta_k(G^*, g^*), G^* \in \mathcal{L}(\mathbb{R}^n)^*,$

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$g^* \in (\mathbb{R}^n)^*$  ( $*$  denotes the dual space) satisfies:

$$\eta_k(G^*, g^*) \leq L \|s_k\| \left\{ \frac{1}{3} \|s_k\|^2 \|g^*\|^2 + (g^*, G^* s_k) + \|G^*\|^2 \right\}^{1/2}. \quad (2.12)$$

It is easy to see that an appropriate choice of  $U_k(\cdot)$  in the class defined by (2.5) attains this bound and hence the support function can be computed as:

$$\eta_k(G^*, g^*) = L \|s_k\| \left\{ \left\langle \begin{pmatrix} G^* \\ g^* \end{pmatrix}, Q_k \begin{pmatrix} G^* \\ g^* \end{pmatrix} \right\rangle \right\}^{1/2} \quad (2.13)$$

where  $\langle \cdot, \cdot \rangle$  is the obvious inner product in  $\mathcal{L}(\mathbb{R}^n) \times \mathbb{R}^n$  and  $Q_k$  in the matrix defined from the right hand side of (2.12). We can check that  $Q_k > 0$  (unless  $s_k = 0$ ).

The above discussions may be combined in the following:

Consider the problem of estimating  $G_k$  from

$$G_{k+1} = G_k + V_k \quad (2.14)$$

$$z_k = G_k s_k + w_k, \text{ where } z_k = g_{k+1} - g_k. \quad (2.15)$$

Let  $G_0 \in \Omega_0$  where

$$\Omega_0 = \{G \in \mathcal{L}(\mathbb{R}^n) \mid \langle G - \hat{G}_0, \hat{G}_0^{-1}(G - \hat{G}_0) \rangle_{\mathbb{R}^n} \leq 1\} \quad (2.16)$$

and  $\hat{G}_0, \pi_0 > 0$  are given.

Then

Proposition 1

$$\begin{pmatrix} v_k \\ w_k \end{pmatrix} \in \Omega_k, \text{ where}$$

$$\Omega_k = \{(v) \in \mathcal{L}(\mathbb{R}^n) \times \mathbb{R}^n \mid \frac{1}{L \|s_k\|} \langle v, Q_k^{-1} v \rangle_{\mathcal{L}(\mathbb{R}^n) \times \mathbb{R}^n} \leq 1\}. \quad (2.17)$$

#### 4. SOLUTION OF THE ESTIMATION PROBLEM

The estimation problem can now be solved using the work of BERTSEKAS [1]. It consists of recursively estimating the sets  $\Omega_k$ , which are ellipsoids. The centre of the ellipsoid is the desired estimate. These results are summarised in



Proposition 2

$$\hat{\Omega}_{K+1} = \left\{ G | \langle G - \hat{G}_{K+1}, \Pi_{K+1}^{-1} (G - \hat{G}_{K+1}) \rangle_{\mathbb{R}^{n^2} + \mathbb{R}^n} \leq 1 - \gamma_K \right\} \quad (2.18)$$

where  $\Pi_{K+1}$  satisfies

$$\Pi_{K+1} G = G P_{K+1}, \quad k=0,1,2,\dots \quad (2.19)$$

and  $P_K$  is given by

$$P_{K+1} = (1 + \|s_k\|) \left\{ P_k + L^2 \|s_k\| I_n - \frac{L^2 \|s_k\|}{2} I_n \frac{[P_k + \frac{L^2 \|s_k\|}{2} I_n] s_k s_k' [P_k + \frac{L^2 \|s_k\|}{2} I_n]}{(s_k, [P_k + \frac{L^2 \|s_k\|}{2} I_n] s_k)} \right\} \quad (2.20)$$

$$\hat{G}_{K+1} = \hat{G}_K + \frac{[z_k - \hat{G}_K s_k] s_k' [P_k + \frac{L^2 \|s_k\|}{2} I_n]}{(s_k, [P_k + \frac{L^2 \|s_k\|}{2} I_n] s_k)} \quad (2.21)$$

and

$$v_k = \frac{\|z_k - \hat{G}_K s_k\|^2}{L^2 \|s_k\| [1 + \|s_k\|] s_k' [P_k + \frac{L^2 \|s_k\|}{3} I_n] s_k} \quad (2.22)$$

Proposition 3

If  $H_k = (\hat{G}_k)^{-1}$  exists and  $\hat{G}_{K+1}$  is generated by (2.20) - (2.22) then

$$H_{K+1} = H_k + \frac{[s_k - H_k z_k] d_k' H_k}{\alpha_k + d_k' [s_k - H_k z_k]} \quad (2.23)$$

is the inverse of  $G_{K+1}$ , where

$$\alpha_k = \frac{(s_k, [P_k + \frac{\|s_k\|}{3} I_n] s_k)}{(s_k, [P_k + \frac{\|s_k\|}{2} I_n] s_k)} < 1 \quad (2.24)$$

$$d_k = \frac{[P_k + \frac{\|s_k\|}{2} I_n] s_k}{(s_k, [P_k + \frac{\|s_k\|}{2} I_n] s_k)} \quad (2.25)$$

## 5. NEW ALGORITHMS FOR FUNCTION $g$ MINIMIZATION

Since we are looking for an estimate of the Hessian (or inverse of the Hessian) it is desirable that our estimates are symmetric. This suggests the following algorithm:

- (i) Propagate  $H_k$  and  $P_k$  according to (2.23) and (2.20)
- (ii) Symmetrize  $H_k$  to obtain  $H_k^s$
- (iii) Find the closest approximation  $\hat{H}_k$  to  $H_k^s$  so that the secant equation  $s_k = Xz_k$  is satisfied.
- (iv) The new step is computed according to Powell's dog-leg strategy (cf. POWELL [3])

We now present a number of convergence results corresponding to the use of different estimates for the Hessian.

Suppose we update  $P_k$  and  $H_k$  according to (2.20) and (2.23)-(2.25) with  $P_0 = \sigma^2 I$ . The new step is chosen according to the formula

$$s_k = -H_k g_k, \text{ and let us update } \Pi_k \text{ according to} \quad (5.1)$$

$$\begin{cases} \Pi_{k+1} = (1 + \|s_k\|) \left[ \Pi_k + \|s_k\| I_n - \frac{(2\alpha_k - 1)}{\alpha_k^2} \frac{d_k d_k'}{(s_k' d_k)} \right] \\ \Pi_0 = \sigma^2 I. \end{cases} \quad (5.2)$$

We then have:

### Lemma 5.1

$$\Pi_k \geq P_k \geq 0 \quad \forall k \geq 0. \quad (5.3)$$

We know, that if  $\hat{G}_k$  is non-singular then

$$\hat{G}_{k+1} = \hat{G}_k + \frac{(z_k - \hat{G}_k s_k) d_k'}{\alpha_k}, \quad k=0,1,2,\dots \quad (5.4)$$

We can then show

### Lemma 5.2

There exists a  $\mu > 0$ , such that

$$\|\hat{G}_k - \hat{G}_k\|^2 \leq \mu, \quad k \geq 0. \quad (5.5)$$

These ideas enable us to prove the following basic convergence theorem:



### Theorem 5.3

Let  $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be differentiable in an open convex neighborhood  $D$  of  $x^*$ , where  $x^*$  satisfies  $g(x^*)=0$  and we also have  $Dg(x^*) = G(x^*)$  is non-singular. Let us suppose that  $G(\cdot)$  satisfies

$$\|G(x) - G(y)\| \leq L\|x - y\|, \forall x, y \in D. \quad (5.6)$$

Then for each  $\gamma \geq 0$ ,  $r \in [0, 1]$ ,  $\exists \delta = \delta(\gamma, r)$ ,  $\epsilon = \epsilon(\gamma, r)$  such that if  $\|x_0 - x^*\| \leq \delta$  and  $\|\hat{G}_0 - G_0\| \leq \gamma\sigma$ ,  $\sigma \in [0, \epsilon]$ , then, the sequence

$$x_{k+1} = x_k - [\hat{G}_k]^{-1} g_k \quad (5.7)$$

converges to  $x^*$ .

Moreover

$$\|x_{k+1} - x^*\| \leq \gamma \|x_k - x^*\| \text{ and the sequence} \quad (5.8)$$

$(\|\hat{G}_k\|)_{k=0,1,\dots}$  and  $(\|\hat{H}_k\|)_{k=0,1,\dots}$  are uniformly bounded.

Theorem 5.3 shows that we obtain linear convergence. One can show that the convergence is actually superlinear.

So far we have constructed an algorithm which uses the output of the filter directly. As we have previously remarked it would be desirable to "symmetrize" the estimate and use this as in the algorithm. It can be shown that an algorithm using the symmetrized estimate converges linearly under the same hypotheses as that of Theorem 5.3. However a proof of convergence of the algorithm when the estimates are also chosen to satisfy the secant equation is at present not available. The details of the proof of the various results, presented in this paper will appear elsewhere [cf. MITTER-TOLDALAGI [2] ].

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Acknowledgement

The research of the first author has been supported by the Air Force Office of Scientific Research under Grant No. AFOSR 77-3281.